Simplified equations for natural frequencies of pipes on elastic foundation conveying gases

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Abstract—Pipelines are commonly subjected to harmonic forces induced by reciprocating centrifugal pumps and compressors, unbalanced moments and forces in turbomachinery, pressure surge, and momentum changes due to valve operation. To guard resonance and fatigue failure, the natural frequencies of pipelines need to be accurately characterized. The present study introduces two sets of simplified equations to compute the natural frequencies of pipes conveying gases resting on uniform elastic foundation for different boundary conditions. While the first set is analytically derived by solving the differential equation, the second set is obtained based on a decoupling approach. A finite element code is built to verify the results computed using both sets of equations and excellent agreement is obtained.

Keywords—natural frequency, pipes conveying fluids, simplified equations, analytical solution, finite element, Winkler foundation

I. INTRODUCTION

Several engineering fields depend on pipelines to convey their products (e.g., oil, gases, chemicals, water) from centers of production to centers of consumption. Hence, an efficient analysis of the dynamic response of pipes conveying fluids considering fluid-structure interactions is necessary for an optimal design. Free vibration analysis of pipelines conveying fluids is essential to identify the conditions of instability and their associated modes, avoid resonance phenomena, and prolong fatigue life. Various studies investigate the free vibration analysis of pipes conveying fluids. Long [1] experimentally studied the dynamic characteristics of such pipes showing that an increase of the fluid velocity causes a decrease of the natural frequencies. Benjamin [2] introduced a theoretical study using the principle of virtual work to investigate the instability of articulated pipes. Gregory and Paidoussis [3] extended the work in [2] for flexible pipes by deriving the governing differential equation using the Newtonian mechanics. Their analysis showed that the flutter is the instability mode associated with the cantilever pipes introducing exact and approximate solutions for obtaining natural frequencies and the critical velocity. In review articles, Paidoussis [4-5] discussed instabilities of cylindrical structures for different classes of problems. Yi-min et al. [6] used the Galerkin method to compute natural frequencies and critical velocities for different boundary conditions. Several numerical solutions were adopted to analyze the dynamic response of pipes conveying fluids such as the differential quadrature method [7], isogeometric analysis [8]. He’s variational iteration method [9], etc. Moreover, the effect of the soil was investigated. For instance, Roth [10] derived simple formulas to compute the critical velocity of pinned-pinned and fixed-fixed pipes resting on elastic foundations. A finite element analysis [11] was adopted to analyze cantilever pipes resting on nonuniform elastic foundations. Li and Hu [12] obtained the critical velocity of fluid-conveying magneto-electro-elastic pipe resting on Winkler foundations based on the Timoshenko beam theory.

Fluids conveyed in pipes include liquids and gases. Compared to the mass of steel pipes, the liquid mass is considerable and cannot be ignored. Conversely, the gas mass is negligible compared to that of steel pipes. Most past research incorporated the self-weight of the fluid resulting in complex governing equations that were solved numerically. Conversely, neglecting the mass of the fluids, as may be justified for pipes conveying gases, makes an analytical solution attainable. Hence, the present article introduces simplified equations to compute the natural frequencies and the critical velocity of pipes conveying gases resting on uniform elastic foundation for various boundary conditions using two techniques (i.e., an analytical solution and a decoupling approach).

II. FORMULATION

A. Analytical approach

The differential equation of equilibrium for a pipe resting on uniform elastic foundation conveying fluid [13] (Fig.1) is

\[
EI \left( \frac{\partial^4 v}{\partial x^4} \right) + \left( m_p + m_f \right) \left( \frac{\partial^2 v}{\partial t^2} \right) + 2m_f u_f \left( \frac{\partial^2 v}{\partial x \partial t} \right) + m_f u_f^2 \left( \frac{\partial^2 v}{\partial x^2} \right) + K v = 0
\]

(1)

in which \( v = v(x,t) \) is the vertical displacement at the pipe centerline in which \( x \) denotes a longitudinal coordinate along the pipe, \( t \) is the time, \( EI \) is the bending rigidity of the pipe, \( m_p \) is the pipe mass per unit length, \( m_f \) is the fluid mass per unit length, \( u_f \) is the fluid velocity in the longitudinal direction taken as constant in the present formulation and \( K \) is the uniform elastic foundation stiffness. Equation (1) can be re-written in a dimensionless form as
\[ (\partial^{2} \psi / \partial x^{2}) + (\partial^{2} \psi / \partial t^{2}) + 2 \beta^{0.5} \gamma (\partial^{2} \psi / \partial x \partial t) + \gamma^{2} (\partial^{2} \psi / \partial x^{2}) + \widetilde{K} \psi = 0 \]  
\tag{2}

where \( \psi = x/L, \quad \nu = v/L, \quad \gamma = \sqrt{m_{p} / EI}, \quad \widetilde{K} = KL^{4}/EI, \quad T = t / \left( (m_{p} + m_{r})L^{4}/EI \right). \) \( \beta = m_{r}/(m_{p} + m_{r}) \) and \( L \) is the length of the pipe. The mass of gases is very small compared to that of the steel pipe. Hence, parameter \( \beta \) can be neglected in (2) yielding

\[ (\partial^{2} \psi / \partial x^{2}) + (\partial^{2} \psi / \partial t^{2}) + \gamma^{2} (\partial^{2} \psi / \partial x^{2}) + \widetilde{K} \psi = 0 \]  
\tag{3}

The solution of the dimensionless vertical displacement \( \widetilde{v}(\widetilde{x}, \widetilde{t}) \) takes the form

\[ \widetilde{v}(\widetilde{x}, \widetilde{t}) = A e^{i\alpha \widetilde{t}} e^{i\alpha \widetilde{x}} \]  
\tag{4}

where \( \alpha = \alpha \) and \( \alpha = \alpha \sqrt{(m_{p} + m_{r})}L^{4}/EI \). By substituting in (3), one obtains the characteristic equation

\[ \alpha^{4} - \gamma^{2} \alpha^{2} + (\alpha^{2} - \widetilde{K}) = 0 \]  
\tag{5}

The four roots of this quartic equation are

\[ \alpha = \pm \sqrt{\gamma^{2} \pm \sqrt{\gamma^{4} - 4 \widetilde{K} + 4 \alpha^{2}}} \]  
\tag{6}

\[ \text{Fig.1. A pipe resting on uniform elastic foundation conveying fluid} \]

Three boundary condition (BC) cases are considered in the present study:

1- Pinned-Pinned Case (P-P):

The boundary conditions are \( \widetilde{v} = \partial \widetilde{v} / \partial \widetilde{x} = 0 \) at \( \widetilde{x} = 0 \) and \( \widetilde{x} = 1 \). By substituting in (4), one obtains the system of equations

\[ \begin{bmatrix} 1 & 1 & 1 & 1 \\ \alpha_0 & \alpha_0 & \alpha_0 & \alpha_0 \\ e^{\alpha_0} & e^{\alpha_0} & e^{\alpha_0} & e^{\alpha_0} \\ \alpha_0 e^{\alpha_0} & \alpha_0 e^{\alpha_0} & \alpha_0 e^{\alpha_0} & \alpha_0 e^{\alpha_0} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = 0 \]  
\tag{7}

By setting the determinant of the matrix of coefficients to zero, the nontrivial solution must satisfy the characteristic equation

\[ Y \sin \left( \sqrt{\gamma^{2} - \alpha^{2}} \right) \sin \left( \sqrt{\gamma^{2} + \alpha^{2}} \right) = 0 \]  
\tag{8}

where \( Y = \gamma^{2} - 4 \widetilde{K} + 4 \alpha^{2} \), which yields the explicit formula for the dimensionless natural frequencies \( \alpha \).

\[ \alpha = \sqrt{\alpha^{2} - 4 \widetilde{K} + 4 \alpha^{2}} \]  
\tag{9}

in which \( r = 1, 2, 3, ... \) is the mode number. The corresponding dimensionless critical velocities \( \gamma_{cr} \) are obtained by setting \( \alpha = 0 \) in (9) yielding

\[ \gamma_{cr} = \sqrt{\alpha^{2} - \gamma^{2}} \]  
\tag{10}

Equation (10) is identical to that reported in [10].

2- Fixed-Pinned Case (F-P):

The boundary conditions for this case are \( \widetilde{v} = \partial \widetilde{v} / \partial \widetilde{x} = 0 \) at \( \widetilde{x} = 0 \) and \( \widetilde{v} = \partial \widetilde{v} / \partial \widetilde{x} = 0 \) at \( \widetilde{x} = 1 \). By substituting in (4), one obtains

By setting the determinant of the matrix of coefficients to zero, one recovers the characteristic equation

\[ \sqrt{\gamma} \sin \left( 2 \sqrt{\gamma} \right) - \sqrt{\gamma} \sin \left( 2 \sqrt{\gamma} \right) = 0 \]  
\tag{12}

in which \( \gamma \) and \( \gamma \) is iteratively solved for the natural frequencies \( \alpha \). Again, the corresponding dimensionless critical velocities \( \gamma_{cr} \) are obtained by setting \( \alpha = 0 \) in (12).

3- Fixed-Fixed Case (F-F):

The boundary conditions for this case are \( \widetilde{v} = \partial \widetilde{v} / \partial \widetilde{x} = 0 \) at \( \widetilde{x} = 0 \) and \( \widetilde{x} = 1 \). Again, by substituting in (4), one obtains

By setting the determinant of the matrix of coefficients to zero, the characteristic equation is recovered

\[ \gamma \sin \left( \gamma \sqrt{X^{2} - \gamma^{2}} \right) = 0 \]  
\tag{14}

where \( X, Y = 0.25 \gamma^{2} \pm 0.5 \sqrt{\alpha^{2} - \gamma^{2}} \). Given parameters \( \gamma \) and \( \gamma \) is iteratively solved for the natural frequencies \( \alpha \). Again, the corresponding dimensionless critical velocities \( \gamma_{cr} \) are computed by setting \( \gamma = 0 \) in (14).

**B. Decoupling approach**

The proposed approach is akin to that presented in [6]. In contrast to [6] which omits the soil effect and adopts a Galerkin type of solution, the present solution is based on the analytical solution derived in the previous section and accounts for the soil effect. The present solution also provides insight on the relationship between dimensionless natural frequencies and dimensionless critical velocities. The dimensionless natural frequency is assumed to take the form
\[ \omega_r = H(\bar{K}, \bar{\alpha}_r) \times G(\gamma, \gamma_r) \]  \hspace{1cm} (15)

in which \( H(\bar{K}, \bar{\alpha}_r) \) represents the natural frequency of a beam on elastic foundation in the absence of gas and is a function of the soil stiffness \( \bar{K} \) and the boundary conditions for the case under consideration, characterized by \( \bar{\alpha}_r \), while \( G(\gamma, \gamma_r) \) refers to the effect of the gas velocity and hence is a function of the dimensionless velocity \( \gamma \) and critical velocity \( \gamma_r \). Function \( H(\bar{K}, \bar{\alpha}_r) \) is obtained by ignoring the term \( \gamma^2 \bar{\alpha}^2 \) in (5) yielding

\[ \bar{\alpha}_r^2 - (\bar{\alpha}_r^2 - \bar{K}) = 0 \]  \hspace{1cm} (16)

or

\[ \bar{\alpha}_r = \sqrt{\bar{K} + \bar{\alpha}_r^2} \]  \hspace{1cm} (17)

where \( \bar{\alpha}_r \) is computed from the solution of the differential equation for flexural vibration of a beam in the absence of elastic foundation. The characteristic equations corresponding to the P-P, F-P, and F-F boundary conditions and their solutions [14] are summarized in TABLE I.

<table>
<thead>
<tr>
<th>BCs</th>
<th>Characteristic equation</th>
<th>( \bar{\alpha}_r ) for first three modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>P-P</td>
<td>( \sin \bar{\alpha}_r = 0 )</td>
<td>( \bar{\alpha}_r = r\pi ), ( r = 1, 2, 3, \ldots )</td>
</tr>
<tr>
<td>F-P</td>
<td>( \tan \bar{\alpha}_r = \tanh \bar{\alpha}_r )</td>
<td>( \bar{\alpha}_r = 3.9266 ), ( \bar{\alpha}_r = 7.0685 ), ( \bar{\alpha}_r = 10.21 )</td>
</tr>
<tr>
<td>F-F</td>
<td>( \cos \bar{\alpha}_r \cosh \bar{\alpha}_r = 1 )</td>
<td>( \bar{\alpha}_r = 4.7300 ), ( \bar{\alpha}_r = 7.8532 ), ( \bar{\alpha}_r = 10.995 )</td>
</tr>
</tbody>
</table>

In order to obtain function \( G(\gamma, \gamma_r) \) for the P-P case, (9) is re-written in terms of the dimensionless critical velocity formula (10) as

\[ \bar{\omega}_r = \sqrt{\bar{K} + (r\pi)^2} \sqrt{1 - (\gamma/\gamma_r)^2} \]  \hspace{1cm} (18)

By comparing (15), (17), and (18), one concludes that the function \( G(\gamma, \gamma_r) \) for P-P case is exactly

\[ G(\gamma, \gamma_r) = \sqrt{1 - (\gamma/\gamma_r)^2} \]  \hspace{1cm} (19)

where \( \gamma_r \) is the dimensionless critical velocity at mode \( r \) as computed from (10). By analogy, the form of the function \( G(\gamma, \gamma_r) \) introduced in (19) is assumed to hold true for the other two BCs (F-P and F-F) while changing the equation for the dimensionless critical velocity \( \gamma_r \) to correspond to the F-P and F-F cases by setting \( \bar{\alpha}_r = 0 \) in (12) and (14), respectively. It is noted that \( G(\gamma, \gamma_r) \) is exact for the P-P case but approximate for the other cases. The effect of this approximation is assessed in the comparisons provided in Section III.

C. Finite Element approach

The value of parameter \( \beta \) does not exactly vanish for gases. Instead, it takes a very small value within range \( 0 < \beta \leq 0.01 \) for most pipes conveying gases. In order to verify the validity of the above analytical and decoupling approaches which take \( \beta = 0 \), a finite element (FE) solution that considers the effect of fluid mass (\( \beta \neq 0 \)), is adopted to numerically solve (2). The FE solution is built based on the variational form of Hamilton’s principle which takes the form

\[ \int_0^L \delta(T - U)dt = 0 \]

\[ T = 0.5 \int_0^L m_r \dot{v}^2 dx + 0.5 \int_0^L m_r \left[ u_r + (\dot{v} + u_r \gamma)^2 \right] dx \]  \hspace{1cm} (20)a-c

\[ U = 0.5 \int_0^L E I \dot{v}^2 dx + 0.5 \int_0^L K \dot{v}^2 dx \]

where \( T \) is the kinetic energy, \( U \) is the internal strain energy, and all primes denote derivatives with respect to coordinate \( x \) and over dots denote time derivatives. The FE formulation is based on a beam element with two end nodes with two degrees of freedom per node (i.e., transverse translation and rotation). The vertical displacement \( v = H^T(x) d \) through Hermite polynomial functions \( H(x) \) and \( \omega \) is the natural frequency. Substitution into the variational principle yields

\[ (K + i\omega C - \omega^2 M)d = 0 \]  \hspace{1cm} (21)

where \( K \) is the stiffness matrix of the pipe and the soil, \( C \) is the damping matrix, and \( M \) is the mass matrix. More details about the FE formulation and the matrices appearing in (21) can be found in [11]. The results of the FE solution when \( \beta = 0.2 \) and 0.5 are compared to those published in the literature and found that 18 elements are adequate to accurately predict the natural frequencies.

III. VERIFICATION EXAMPLES

This section presents two groups of verification examples: (1) Comparisons between the results obtained from analytical, decoupling, and FE approaches when the mass ratio is set to zero (\( \beta = 0 \)) showing the limitations of the decoupling approach, and (2) Comparisons between the results computed from analytical and FE methods while setting parameter \( \beta \) to 0.01.

A. Group 1: Gas to Pipe Mass Ratio = 0.00

A parametric study is carried out to compare dimensionless natural frequencies and critical velocities predicted from the analytical and decoupling approaches with those based on the FE solutions under various dimensionless soil stiffness. Figs. 2-4 show the dimensionless natural frequency distributions for the first three eigenvalues versus the dimensionless velocity for the P-P, F-P, and F-F cases for a dimensionless soil stiffness \( \bar{K} = 700 \). Fig.2 shows that the dimensionless critical velocity is 2.4\( \pi \) and is associated with the second mode while dimensionless critical velocities in Figs. 3 and 4 are associated with the first mode and equal to 2.57\( \pi \) and 2.99\( \pi \), respectively. Excellent agreement is observed between analytical and FE solutions for all three cases considered. While the predictions of the decoupling solution are identical to those of analytical and FE solutions for the P-P case, it leads to approximate estimates for the other two cases given the approximate nature of function \( G(\gamma, \gamma_r) \) in (19). TABLE II presents first natural frequency predictions obtained from analytical and decoupling
solutions. While the percentage error for the F-P case is less than 4.5% when $\gamma \leq 1.5\pi$ and $K \leq 700$, it is less than 5% when $\gamma \leq 2.5\pi$ and $K \leq 700$ for the F-F case.

**TABLE II. Percentage error of first natural frequency predictions**

<table>
<thead>
<tr>
<th>BCs</th>
<th>soil stiffness $K$</th>
<th>velocity $\gamma$</th>
<th>1st frequency $(\omega_1)$</th>
<th>Percentage Error $^a$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Analytical</td>
<td>Decoupling</td>
</tr>
<tr>
<td>F-P</td>
<td>0</td>
<td>$2\pi$</td>
<td>$0 \pm 15.68\iota$</td>
<td>$0 \pm 15.07\iota$</td>
</tr>
<tr>
<td></td>
<td>700</td>
<td>$1.5\pi$</td>
<td>$25.99+0\iota$</td>
<td>$24.84+0\iota$</td>
</tr>
<tr>
<td>F-F</td>
<td>0</td>
<td>$2.5\pi$</td>
<td>$0 \pm 17.36\iota$</td>
<td>$0 \pm 16.78\iota$</td>
</tr>
<tr>
<td></td>
<td>700</td>
<td>$2.5\pi$</td>
<td>$19.79+0\iota$</td>
<td>$18.84+0\iota$</td>
</tr>
</tbody>
</table>

*Percentage error $^a = (\text{Analytical solution} - \text{Decoupling solution}) \times 100 / \text{Analytical solution}*$

**B. Group 2: Gas to Pipe Mass ratio = 0.01**

To assess the validity of the equations derived for pipes conveying gases, FE solutions are obtained at $0.01 \beta = 0$ and compared to those based on the analytical approach for various soil stiffness. Figs. 5-7 show dimensionless natural frequencies of the first two modes versus the dimensionless velocity for the P-P, F-P, and F-F cases at various dimensionless soil stiffness values $K = 0, 400, 700$. As the soil stiffness increases, the dimensionless natural frequencies and critical velocities increase. The critical velocity is not necessarily associated with the first mode but depends on the soil stiffness and boundary conditions. Excellent agreement is observed between the predictions of the analytical approach with $\beta = 0$ and those of the FE solution with $\beta = 0.01$ which demonstrates the validity of the assumption $\beta = 0$ forming the basis of the analytical and decoupling approaches.

**IV. SUMMARY**

The present paper introduces accurate and simple mathematical formulas to compute the dimensionless natural frequencies and critical velocities for pipes conveying gases resting on uniform elastic foundations. Two approaches are adopted to derive these formulas: (1) Analytical approach which presents a solution of the governing differential equation, and (2) Decoupling approach which decouples the
fluid effect in a separate term/function. While the predictions of the natural frequency based on the analytical solutions are in an excellent agreement with those obtained from the FE solutions when the mass ratio is less than 0.01 for all three cases considered, the predictions of the decoupling solution depends on the boundary conditions. Decoupling solution predictions are identical to those of analytical and FE solutions for the P-P case and approximate for the F-P and F-F cases due to the approximate nature of function \( G(\gamma, \gamma) \), which represents the fluid effect, for these cases. The percentage error in the predictions of the decoupling solution is less than 5% when the dimensionless velocity \( \gamma \leq 1.5\pi \) for the F-P case and \( \gamma \leq 2.5\pi \) for the F-F case given that the dimensionless soil stiffness \( K \leq 700 \). It is recommended to analytically incorporate the effect of fluid mass.

**Fig. 6.** Real (Re) and imaginary (Im) components of the dimensionless natural frequency for first two eigenvalues versus the dimensionless velocity for the F-P case (While positive component of the imaginary component of the dimensionless natural frequency is shown for the first mode, the negative component is shown for the second mode for clarity)

**Fig. 7.** Real (Re) and imaginary (Im) components of the dimensionless natural frequency for first two eigenvalues versus the dimensionless velocity for the F-F case (While positive component of the imaginary component of the dimensionless natural frequency is shown for the first mode, the negative component is shown for the second mode for clarity)

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**REFERENCES**


